

THE KERNEL OF THE DETERMINANT MAP ON PURE C^* -ALGEBRAS

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ABSTRACT. In a simple C^* -algebra with suitable regularity properties, any unitary or invertible element with de la Harpe–Skandalis determinant zero is a finite product of commutators.

1. INTRODUCTION

The notion of determinant has been, as might be expected, generalized to and investigated in the realm of operator algebras. Fuglede and Kadison defined a determinant on a type II_1 factor ([FK52]) and used it to settle fundamental questions on the trace of the factor. Subsequently, the Fuglede–Kadison determinant has found manifold applications (see [dlH13] and references therein). De la Harpe and Skandalis generalized the Fuglede–Kadison determinant to arbitrary Banach algebras ([dlHS84a]), and it played a role in their investigations on the basic structure of the unitary and general linear groups of a C^* -algebra ([dlHS84a], [dlHS84b], [dlHS85]). Again, the de la Harpe–Skandalis determinant has led to subsequent work (e.g., [Tho93], [Ng12]). It has been enormously useful in the Elliott classification program, not least because of its connections with uniqueness theorems and the classification of automorphisms (e.g., [KM08], [Lin11]).

In [dlHS84b], de la Harpe and Skandalis show that every invertible or unitary in the kernel of the determinant map of a simple unital AF C^* -algebra is expressible as a finite product of commutators. This type of result had been previously considered for the Fuglede–Kadison determinant ([FdlH80]). Thomsen extended de la Harpe and Skandalis’s result to certain simple slow dimension growth AH C^* -algebras [Tho93]. The first named author extended further these results to simple \mathcal{Z} -stable TAI and simple real rank zero C^* -algebras with strict comparison ([Ng12]). A common characteristic in these results is the abundance of projections in the C^* -algebra. In this paper, we generalize these previous results to a large class of simple C^* -algebras, including those considered by the Elliott classification program, as well as non-nuclear examples.

Let us state the main result of the paper. Let A be a C^* -algebra. Let $U^0(A)$ and $GL^0(A)$ denote the connected components of the unitary and general linear groups of A , respectively. Let Δ_{Tr} denote the universal de la Harpe-Skandalis determinant defined on $GL^0(A)$ (see Section 2 below and [dlHS84a]). Throughout the paper we have focused mostly on the restriction of Δ_{Tr} to $U^0(A)$. However, our methods give analogous results for the kernel Δ_{Tr} on $GL^0(A)$ (see Theorem 6.7).

We say that the C^* -algebra A is pure if the Cuntz semigroup of A is almost unperforated and almost divisible ([Win12, Definition 2.6]). C^* -algebras that tensorially absorb the Jiang-Su C^* -algebra are pure ([Rør04]), but pure C^* -algebras may even be tensorially prime (e.g., the reduced C^* -algebra of the free group on infinitely many generators $C_r^*(\mathbb{F}_\infty)$; see [Rob12, Proposition 6.3.2]).

Theorem 1.1. *Let A be a separable, simple, pure C^* -algebra with stable rank 1 and such that every 2-quasitracial state on A is a trace. Let $u \in U^0(A)$ be such that $\Delta_{\text{Tr}}(u) = 0$ and u is a product of k exponentials; i.e., $u = \prod_{j=1}^k e^{ih_j}$ with $h_j \in A_{\text{sa}}$ for all $j = 1, \dots, k$. Then there exist $7k + 29$ unitaries $v_i, w_i \in U^0(A)$, with $i = 1, \dots, 7k + 29$, such that*

$$u = \prod_{i=1}^{7k+29} (v_i, w_i).$$

A simple corollary of this theorem is that for a C^* -algebra A as in the theorem we have

$$\ker \Delta_{\text{Tr}} \cap U^0(A) = \text{DU}^0(A) = \text{DU}(A),$$

(Corollary 6.6). If we additionally know that the C^* -algebra A has finite exponential rank, then Theorem 1.1 gives a uniform bound on the number of commutators. Lin has shown in [Lin13] that unital, \mathcal{Z} -stable, rationally TAI C^* -algebras have exponential rank $1 + \varepsilon$, so a uniform bound (of 43 commutators) is obtained in this case.

Another motivation for the results of this paper is to test out, extend, and provide a new forum for recent techniques from classification theory. In the past five years, the Elliott classification program has made great strides, resulting in many new tools and ideas which should have wide applicability (see [ET08]). Three regularity properties have come to the foreground: tensorial absorption of the Jiang-Su algebra, nuclear dimension, and pureness (for simple C^* -algebras, the weakest of the three). Pure C^* -algebras form a class of C^* -algebras with a well behaved “comparison theory” – in analogy with the comparison theory of factors, and in the spirit of Blackadar’s paper [Bla88]. An example of a pure C^* -algebra to which Theorem 1.1 applies is $C_r^*(\mathbb{F}_\infty)$ – a non-nuclear tensorially prime C^* -algebra of much interest. Along the way

towards the proof of Theorem 1.1, we also investigate the kernel of the determinant map on C*-algebras of nuclear dimension at most one.

Let us discuss briefly the methods used to prove Theorem 1.1: We use that it is possible to find a special simple C*-algebra B , expressible as an inductive limit of 1-dimensional NCCW complexes, inside a C*-algebra A satisfying the hypotheses of Theorem 1.1. This relies on the classification result from [Rob12]. The approximation of unitaries in $U^0(A) \cap \ker \Delta_{\text{Tr}}$ by products of commutators is then reduced to unitaries in $U^0(B)$. To approximate unitaries in $U^0(B) \cap \ker \Delta_{\text{Tr}}$ by products of commutators, we exploit the fact that it has nuclear dimension at most one. Finally, in order to turn the approximation by products commutators into an exact product of commutators, we use a technique originally used by Fack for AF C*-algebras ([Fac82]), and which we have adapted to a setting without abundance of projections.

2. DEFINITIONS, NOTATION, PRELIMINARY RESULTS

Let A be a C*-algebra. We will denote by A_{sa} the subset of selfadjoint elements of A and by A_+ the positive elements.

Given $a \in A_+$, we let $\text{her}(a)$ denote the hereditary C*-subalgebra generated by a ; i.e., \overline{aAa} . Some times we will write $\text{her}^A(e)$ if the ambient C*-algebra needs to be emphasized.

The unitization of A will be denoted by A^\sim . The unitary group of A will be denoted by $U(A)$ and the connected component of the identity by $U^0(A)$. We will often deal with unitary groups of non-unital C*-algebras. In this case we will let $U^0(A)$ stand for the kernel of the group homomorphism $U^0(A^\sim) \rightarrow U(\mathbb{C})$ induced by the quotient map $A^\sim \rightarrow \mathbb{C}$ (i.e. unitaries of the form $1 + x$, with $x \in A$).

If u and v are elements in $U^0(A)$ (or more generally a group) we denote by (u, v) the commutator $uvu^{-1}v^{-1}$.

Let $A_q = A/[A, A]$ and let us denote by $\text{Tr}: A \rightarrow A_q$ the quotient map (which we call the universal trace on A). By a theorem of Cuntz and Pedersen, if $h \in A_{\text{sa}}$ and $\text{Tr}(h) = 0$ then $h = \sum_{i=1}^{\infty} [x_i^*, x_i]$ for some $x_i \in A$ (see [CP79]). We will often write $a \sim_{\text{Tr}} b$ meaning that $\text{Tr}(a - b) = 0$. We extend Tr to $\bigcup_{n=1}^{\infty} M_n(A)$ by setting $\text{Tr}((a_{i,j})_{i,j=1}^n) = \text{Tr}(\sum_{i=1}^n a_{i,i})$.

We will need the following lemma:

Lemma 2.1. *Let A be a C*-algebra and I a closed two-sided ideal of A .*

- (i) *For any $h \in (A/I)_{\text{sa}}$ such that $h \sim_{\text{Tr}} 0$ and $\varepsilon > 0$ there exists a lift $h' \in A_{\text{sa}}$ of h such that $h' \sim_{\text{Tr}} 0$ and $\|h'\| \leq (1 + \varepsilon)\|h\|$.*
- (ii) *For any $a, b \in A/I$ such that $a \sim_{\text{Tr}} b$ and $a' \in A$, lift of a , there exists $b' \in A$, lift of b , such that $a' \sim_{\text{Tr}} b'$.*

Proof. (i) We first look at the case of that h is finite sum of commutators. Say $h = \sum_{i=1}^n [x_i^*, x_i]$. By lifting each commutator x_i to some $\tilde{x}_i \in A$, we find $\tilde{h} = \sum_{i=1}^n [\tilde{x}_i^*, \tilde{x}_i]$, lift of h , such that $\tilde{h} \sim 0$. Let $(e_\lambda)_\lambda$ be an approximately central approximate unit of the ideal I . Then $(1 - e_\lambda)\tilde{h}(1 - e_\lambda)$ is also lift of h and $\|(1 - e_\lambda)\tilde{h}(1 - e_\lambda)\| \rightarrow \|h\|$. These elements are almost sums commutator:

$$(1 - e_\lambda)\tilde{h}(1 - e_\lambda) - \sum_{i=1}^n [((1 - e_\lambda)\tilde{x}_i)^*, (1 - e_\lambda)\tilde{x}_i] \rightarrow 0.$$

So for some λ we find that $h' = \sum_{i=1}^n [((1 - e_\lambda)\tilde{x}_i)^*, (1 - e_\lambda)\tilde{x}_i]$ is the desired lift of h .

In the general case, we can write $h = \sum_{n=1}^\infty h_i$, where h_i is a finite sum of commutators for all i , $\|h_1\| \leq (1 + \frac{\varepsilon}{2})\|h\|$ and $\|h_i\| \leq \frac{1}{2^i}$ for $i \geq 2$. We then choose lifts for each of the h_i 's and we are done.

(ii) Let $c = b - a$. By (i) we can find $c' \in A$, lift of c , such that $c' \sim_{\text{Tr}} 0$. Then $b' = a' + c'$ is the desired lift of b . \square

We will denote by $\Delta_{\text{Tr}}: \text{GL}_\infty^0(A) \rightarrow A_q/\text{Tr}(K_0(A))$ the de la Harpe-Skandalis determinant. This is the group homomorphism such that $\Delta_{\text{Tr}}(e^{2\pi i y})$ is equal to the equivalence class of $\text{Tr}(y)$ in $A_q/\text{Tr}(K_0(A))$ for all $y \in M_\infty(A)$. We will, however, be mostly concerned with the restriction of Δ_{Tr} to $U^0(A)$, and in particular the kernel of this restriction.

The following lemmas will be used repeatedly throughout the paper:

Lemma 2.2. *Let $M > 0$ and $\varepsilon > 0$. There exists $\delta > 0$ such that if $a, b \in A_{\text{sa}}$ are such that $\|a\|, \|b\| \leq M$ and $\|[a, b]\| < \delta$ then*

$$e^{i(a+b)} = e^{ia}e^{ib}e^{ic},$$

where $c \sim_{\text{Tr}} 0$ and $\|c\| < \varepsilon$.

Proof. For each $r > 0$ there exists $\delta > 0$ such that $\|[a, b]\| < \delta$ and $\|a\|, \|b\| < M$ imply $\|e^{a+b}e^{-a}e^{-b} - 1\| < r$. (Assume, to the contrary, that no such δ_0 exists. Then for each $n \geq 1$ there exist a C^* -algebra A_n and $a_n, b_n \in A_n$ with $\|a_n\|, \|b_n\| \leq M$, $\|[a_n, b_n]\| < 1/n$, and $\|e^{a_n+b_n}e^{-a_n}e^{-b_n} - 1\| \geq r$. Consider the C^* -algebra $A = \prod_{n=1}^\infty A_n / \bigoplus_{n=1}^\infty A_n$ and the elements $a, b \in A$ that lift to $(a_n)_{n=1}^\infty$ and $(b_n)_{n=1}^\infty$ respectively. Then a and b commute, and so $e^{a+b} = e^a e^b$. Hence $\|e^{a_n+b_n}e^{-a_n}e^{-b_n} - 1\| \rightarrow 0$, which contradicts that $\|e^{a_n+b_n}e^{-a_n}e^{-b_n} - 1\| \geq r$ for all $n \in \mathbb{N}$.) So let us choose $0 < \delta$ such that $\|e^{i(a+b)}e^{-ia}e^{-ib} - 1\| < 1$, and furthermore $\|\text{Log}(e^{i(a+b)}e^{-ia}e^{-ib})\| < \varepsilon$, whenever $a, b \in A_{\text{sa}}$ are such that $\|[a, b]\| < \delta$ and $\|a\|, \|b\| \leq M$. Set $\text{Log}(e^{i(a+b)}e^{-ia}e^{-ib}) = c$, so that $e^{i(a+b)} = e^{ia}e^{ib}e^{ic}$. It remains to show that $c \sim_{\text{Tr}} 0$. Let $\eta(t) = e^{it(a+b)}e^{-ita}e^{-itb}$ for $t \in [0, 2\pi]$. Then $\|\eta(t) - 1\| < 1$ for all t . So

$$\Delta_{\text{Tr}}(\eta) = \text{Tr}(\text{Log}(\eta(1))) = \frac{1}{2\pi} \text{Tr}(c),$$

by [dlHS84a, Lemma 1]. On the other hand, by [dlHS84a, Lemma 1] again,

$$\begin{aligned}\Delta_{\text{Tr}}(\eta) &= \Delta_{\text{Tr}}(e^{it(a+b)}) - \Delta_{\text{Tr}}(e^{ita}) - \Delta_{\text{Tr}}(e^{itb}) \\ &= \frac{1}{2\pi} \text{Tr}(a+b) - \frac{1}{2\pi} \text{Tr}(a) - \frac{1}{2\pi} \text{Tr}(b) \\ &= 0.\end{aligned}$$

□

Lemma 2.3. *For any $\varepsilon > 0$ and $h \in A_{\text{sa}}$ there exists $\delta > 0$ such that if $\|h' - h\| < \delta$ then $e^{ih} = e^{ih'}e^{ic}$, where $c \sim_{\text{Tr}} h - h'$ and $\|c\| < \varepsilon$.*

Proof. By the previous lemma, we can choose δ such that $e^{ih} = e^{h'}e^{i(h-h')}e^{ic'}$, with $c' \sim_{\text{Tr}} 0$ and $\|c'\|$ small enough. We can do this so that $e^{i(h-h')}e^{ic'} = e^{ic}$, where $c \sim_{\text{Tr}} h - h'$ and $\|c\| < \varepsilon$. □

Lemma 2.4. *Let $x \in A$ be such that $x^2 = 0$. The following are true:*

(i) *For each $\varepsilon > 0$ there exist $v, w \in U^0(A)$ such that*

$$e^{i[x^*, x]} = (v, w)e^{ic},$$

where $c \sim_{\text{Tr}} 0$ and $\|c\| < \varepsilon$.

(ii) *If $\|x\| < \sqrt{\frac{\pi}{2}}$ then there exist $v, w \in U^0(A)$ such that*

$$e^{i[x^*, x]} = (v, w),$$

*with $\|v - 1\|, \|w - 1\| \leq \|e^{ix^*x} - 1\|^{\frac{1}{2}}$.*

Proof. There exists a homomorphism $\phi: M_2(C_0(0, \|x\|)) \rightarrow A$ such that

$$\begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix} \xrightarrow{\phi} x$$

Thus, it suffices to prove the lemma in $B = M_2(C_0(0, \|x\|))$ for $x = \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix}$. Now, since B has stable rank one, x^*x and xx^* are approximately unitarily equivalent in B^\sim . Let us choose $u \in U^0(B)$ (notice that B^\sim has connected unitary group) such that $u^*(x^*x)u$ is close enough to xx^* so that in turn $e^{iu^*(x^*x)u} = e^{ic}e^{ixx^*}$, with $c \sim_{\text{Tr}} 0$ and $\|c\| < \varepsilon$ (applying the previous lemma). Then we find that

$$e^{i[x^*, x]} = e^{ix^*x}e^{-ixx^*} = (e^{ix^*x}, u)e^{ic}.$$

(ii) This follows from [dlHS84b, Lemma 5.13]. □

3. C*-ALGEBRAS OF NUCLEAR DIMENSION AT MOST ONE.

Here we investigate the problem of approximating unitaries $u \in \ker \Delta_{\text{Tr}} \cap U^0(A)$ by products of commutators, when the C*-algebra A has nuclear dimension at most 1. Besides controlling the number of commutators, we pay attention to having multiplicative errors of the form e^{ic} , with $c \sim_{\text{Tr}} 0$. Theorems 3.3 and 3.4 will be used, later on, in the proof of Theorem 1.1.

The reader is referred to [WZ10] for the definition and basic results on the nuclear dimension of C^* -algebras.

Let A be a separable C^* -algebra of nuclear dimension at most $m \in \mathbb{N}$. By [WZ10, Proposition 3.2], there exist completely positive contractive (c.p.c.) order zero maps

$$(3.1) \quad A \xrightarrow{\psi_i} N_i \xrightarrow{\phi_i} A_\infty,$$

with $i = 0, \dots, m$, such that

$$\iota = \sum_{i=0}^m \phi_i \psi_i,$$

where $A_\infty = \prod_{n=1}^\infty A / \bigoplus_{n=1}^\infty A$, $\iota: A \rightarrow A_\infty$ is the inclusion of A as constant sequences, and

$$N_i = \prod_{n=1}^\infty F_{i,n} / \bigoplus_{n=1}^\infty F_{i,n},$$

with $F_{i,n}$ finite dimensional C^* -algebras for all $i = 0, \dots, m$ and $n \in \mathbb{N}$. Furthermore, for each $i = 0, \dots, m$ there exists $e_i \in A' \cap A_\infty$ such that $\phi_i \psi_i(a) = e_i a$ for all $a \in A$. We shall call $\{e_i\}_{i=0}^m$ the partition of unity associated to the decomposition $(\psi_i, N_i, \phi_i)_{i=0}^m$. (In the case that A is unital, $e_i = \phi_i \psi_i(1)$, for all $i = 0, \dots, m$.) In general, the elements of the partition of unity do not commute with each other. Nevertheless, if A is unital and has nuclear dimension at most one, then $e_1 = 1 - e_0$ and so e_0 and e_1 do commute.

Lemma 3.1. *Let $N = \prod_{n=1}^\infty F_n / \bigoplus_{n=1}^\infty F_n$, where each F_n is a finite dimensional C^* -algebra. Let $h \in N_{sa}$ be such that $h \sim_{\text{Tr}} 0$. Then there exist $h_1, h_2 \in N_{sa}$ such that*

- (i) $h = h_1 + h_2$,
- (ii) $[h_1, h_2] = 0$,
- (iii) $h_i = [x_i^*, x_i]$ for some $x_i \in N$ such that $x_i^2 = 0$ and $\|x_i\| \leq \|h\|^{\frac{1}{2}}$, for $i = 1, 2$.

Proof. By [Rob13, Lemma 3.1], we can lift h to $(h_n)_n \in \prod_{n=1}^\infty F_n$ such that $h_n \sim_{\text{Tr}} 0$ for all $n \in \mathbb{N}$. We can then prove the lemma entrywise; i.e., for a finite dimensional C^* -algebra. This case, in turn, reduces at once to the full matrix algebra case. So let us suppose that $h \in M_n(\mathbb{C})_{sa}$ is such that $\text{Tr}(h) = 0$. Conjugating by a unitary, let us further reduce ourselves to the case that h is a diagonal matrix: $h = \text{diag}(\lambda_1, \dots, \lambda_n)$. Finally, conjugating by a permutation, let us also assume that the diagonal entries of h are arranged in such a way that

$$\left| \sum_{i=1}^k \lambda_i \right| \leq \max_{1 \leq i \leq k} |\lambda_i|$$

for all $k = 1, \dots, n$. Let us now proceed, as in the proof of [dlHS84b, Lemma 5.3], by setting $\mu_k = \sum_{i=1}^k \lambda_i$ for all $k = 1, \dots, n$, so that

$$h = \text{diag}(\mu_1, -\mu_1, \mu_3, -\mu_3, \dots) + \text{diag}(\mu_2, -\mu_2, \mu_4, -\mu_4, \dots).$$

The two diagonal matrices on the right hand side are of the form $[x^*, x]$ with $x^2 = 0$ (this follows from the 2×2 case). \square

Theorem 3.2. *Let A be a C*-algebra of nuclear dimension at most 1. Let $h \in A_{\text{sa}}$ be such that $h \sim_{\text{Tr}} 0$. Then for each $\varepsilon > 0$ there exist $h_i \in A_{\text{sa}}$, with $i = 1, 2, 3, 4$ such that*

- (i) $h_1 + h_2 + h_3 + h_4 \approx_\varepsilon h$,
- (ii) $\|[h_1, h_2]\| < \varepsilon$, $\|[h_3, h_4]\| < \varepsilon$, and $\|[h_1 + h_2, h_3 + h_4]\| < \varepsilon$,
- (iii) $h_i = [x_i^*, x_i]$ where $x_i^2 = 0$ and $\|x_i\|^2 \leq \|h\|$ for all $i = 1, 2, 3, 4$.

Proof. Let A be as in the statement. We can reduce ourselves to the case that A is unital. For suppose that, having proved the theorem in the unital case, we have that A is not unital. Then passing to A^\sim and finding $x_1, x_2, x_3, x_4 \in A^\sim$ that satisfy (iii) of the statement of the theorem, we see from $x_i^2 = 0$ that these elements are in fact in A . Also, by [WZ10, Proposition 2.6], there exists a separable C*-algebra $B \subseteq A$ of nuclear dimension at most one, and such that $h \in B$ and $h \sim_{\text{Tr}} 0$ in B . Hence, we can assume that A is separable.

So let us assume that A is unital and separable. Let $e_0, e_1 \in A_\infty$ be a partition of the unit associated to a decomposition $(\psi_i, N_i, \phi_i \mid i = 0, 1)$ (as in (3.1)), so that $\phi_i \psi_i(h) = e_i h$ for $i = 0, 1$. Let us fix $i = 0, 1$. By Lemma 3.1 applied to $\psi_i(h) \in N_i$, there exist selfadjoint elements $\tilde{h}_{i,0}, \tilde{h}_{i,1} \in N_i$ such that

- (1) $\psi_i(h) = \tilde{h}_{i,0} + \tilde{h}_{i,1}$,
- (2) $[\tilde{h}_{i,0}, \tilde{h}_{i,1}] = 0$,
- (3) $\tilde{h}_{i,j} = [\tilde{x}_{i,j}^*, \tilde{x}_{i,j}]$, with $\tilde{x}_{i,j}^2 = 0$ and $\|\tilde{x}_{i,j}\|^2 \leq \|\tilde{h}_{i,j}\|$.

Now composing with ϕ_0 and ϕ_1 we get elements $h_{i,j} := \phi_0(\tilde{h}_{i,j}) \in A_\infty$ that commute (since order zero maps preserve commutation). Let $\pi^{\phi_i}: N_i \otimes \mathbb{C}_0((0, 1]) \rightarrow A_\infty$ be the homomorphism such that $\phi_i(a) = \pi^{\phi_i}(a \otimes t)$ for all $a \in N_i$ (see [WZ10]). Then $x_{i,j} = \pi^{\phi_i}(\tilde{x}_{i,j} \otimes t^{1/2}) \in A_\infty$ satisfies that $h_{i,j} = [x_{i,j}^*, x_{i,j}]$ and $x_{i,j}^2 = 0$. Finally, we can lift the relations $x_{i,j}^2 = 0$ and $\|x_{i,j}\|^2 \leq \|h\|$ to $\prod_{n=1}^\infty A$, which proves the theorem. \square

We get the following consequence of Theorem 3.2:

Theorem 3.3. *Let A be a C*-algebra of nuclear dimension at most 1. Let $h \in A_{\text{sa}}$ be such that $h \sim_{\text{Tr}} 0$. Let $\varepsilon > 0$. Then*

(i) there exist unitaries $v_j, w_j \in U^0(A)$, with $j = 1, 2, 3, 4$, and a selfadjoint $c \in A_{sa}$ such that

$$e^{ih} = \prod_{j=1}^4 (v_j, w_j) \cdot e^{ic},$$

where $c \sim_{\text{Tr}} 0$ and $\|c\| < \varepsilon$.

(ii) If $\|h\| < \frac{\pi}{2}$ then the unitaries from part (i) may be chosen to additionally satisfy that $\|v_j - 1\|, \|w_j - 1\| \leq \|e^{ih} - 1\|^{\frac{1}{2}}$ for $j = 1, 2, 3, 4$.

Proof. Let $h_{j,n} \in A_{sa}$ and $x_{j,n} \in A$, with $j = 1, 2, 3, 4$ and $n \in \mathbb{N}$ be elements that satisfy conditions Theorem 3.2 (i)-(iii) for $\varepsilon = \frac{1}{n}$. Then, for n large enough

$$\begin{aligned} e^{ih} &= e^{i(h_{1,n}+h_{2,n}+h_{3,n}+h_{4,n})} e^{ic_n} \\ &= e^{i(h_{1,n}+h_{2,n})} e^{i(h_{3,n}+h_{4,n})} e^{ic'_n} \\ &= e^{ih_{1,n}} e^{ih_{2,n}} e^{ih_{3,n}} e^{ih_{4,n}} e^{ic''_n}, \end{aligned}$$

where $c_n \sim_{\text{Tr}} c'_n \sim_{\text{Tr}} c''_n \sim_{\text{Tr}} 0$ and $c_n, c'_n, c''_n \rightarrow 0$. Here we have used the asymptotic commutation relations from Theorem 3.2 (ii) in combination with Lemma 2.2, and some algebraic manipulations with exponentials.

To prove (i), we use Lemma 2.4 (i) to find for each $j = 1, 2, 3, 4$ unitaries $v_{j,n}, w_{j,n} \in U^0(A)$ such that

$$e^{ih_{j,n}} = e^{i[(x_{j,n})^*, x_{j,n}]} = (v_{j,n}, w_{j,n}) e^{id_{j,n}},$$

where $d_{j,n} \sim_{\text{Tr}} 0$ and $d_{j,n} \rightarrow 0$ as $n \rightarrow \infty$. To prove (ii), we use Lemma 2.4 (ii) to ensure that $v_{j,n}$ and $w_{j,n}$ can be chosen such that

$$\|v_{j,n} - 1\|, \|w_{j,n} - 1\| \leq \|e^{ih_{j,n}} - 1\|^{\frac{1}{2}} \leq \|e^{ih} - 1\|^{\frac{1}{2}},$$

for all $j = 1, 2, 3, 4$ and $n \in \mathbb{N}$. □

We now investigate products of exponentials in a C^* -algebra of nuclear dimension at most one.

Theorem 3.4. *Let A be a C^* -algebra of nuclear dimension at most 1. Let $u \in U^0(A)$ be of the form $u = e^{ih_1} e^{ih_2} \dots e^{ih_k}$, where $h_j \in A_{sa}$ for $j = 1, \dots, k$ and*

$$\sum_{j=1}^k h_j \sim_{\text{Tr}} 0.$$

Then for each $\varepsilon > 0$ there exist $v_j, w_j \in U^0(A)$, with $j = 1, \dots, 5k-1$ and $c \in A_{sa}$ such that

$$u = \prod_{j=1}^{5k-1} (v_j, w_j) \cdot e^{ic},$$

where $c \sim_{\text{Tr}} 0$ and $\|c\| < \varepsilon$.

Before proving this theorem we need a couple of lemmas:

Lemma 3.5. *Let $N = \prod_{n=1}^{\infty} F_n / \bigoplus_{n=1}^{\infty} F_n$, where F_n is a finite dimensional C*-algebra for all n . Let $h \in N_{sa}$. Then there exist $q, r \in N_{sa}$ such that $h = q + r$, $q \sim_{Tr} 0$, r is in the center of N , and $\|q\| \leq 2\|h\|$, $\|r\| \leq \|h\|$.*

Lemma 3.6. *Let a_1, \dots, a_k and b_1, \dots, b_k be element of a group. We have*

$$\prod_{j=1}^k a_j b_j = \prod_{j=1}^k a_j \cdot \prod_{j=0}^{k-1} b_{k-j} \cdot \prod_{j=1}^{k-1} (v_j, w_j),$$

for some group elements $v_1, w_1, \dots, v_k, w_k$

Proof. Straightforward induction. □

Proof of Theorem 3.4. We will assume that A is separable and unital (as in the proof of Theorem 3.2, the general case reduces to this case).

Let $(\psi_i, N_i, \phi_i \mid i = 0, 1)$ be a factorization of $\iota: A \rightarrow A_{\infty}$ coming from the nuclear dimension of A (as in (3.1)). Let $e_i = \phi_i \psi_i(1)$, with $i = 0, 1$, be the partition of unity associated to it (so $e_0, e_1 \in A' \cap A_{\infty}$ and $e_0 + e_1 = 1$).

Let $h_1, \dots, h_k \in A_{sa}$ be selfadjoint elements as in the statement of the theorem. Since $\sum_{j=1}^k h_j \sim_{Tr} 0$ and the relation \sim_{Tr} is preserved by order zero maps, we have $\sum_{j=1}^k \psi_i(h_j) \sim_{Tr} 0$ for $i = 0, 1$. By Lemma 3.5 applied to each of the $\psi_i(h_j)$ s, we have

$$\psi_i(h_j) = q_{j,i} + r_{j,i},$$

where $q_{j,i} \sim_{Tr} 0$ and $r_{j,i} \in N_i$ is central. Notice that $\sum_{j=1}^k r_{j,i} \sim_{Tr} 0$. By Lemma 3.1 and the fact that the maps ϕ_0 and ϕ_1 are order zero maps, there exist $x_{j,i,0}, x_{j,i,1}, y_{i,0}, y_{i,1} \in A_{\infty}$, for $i = 0, 1$ and $j = 2, \dots, k$ such that

$$\phi_i(\psi_i(h_j)) = [x_{j,i,0}^*, x_{j,i,0}] + [x_{j,i,1}^*, x_{j,i,1}] + \phi_i(r_{i,j})$$

for $i = 0, 1$ and $j = 2, \dots, k$,

$$\phi_i(\psi_i(h_1)) + \sum_{j=2}^k \phi_i(r_{j,i}) = [y_{i,0}^*, y_{i,0}] + [y_{i,1}^*, y_{i,1}],$$

for $i = 0, 1$, and

$$x_{j,i,0}^2 = x_{j,i,1}^2 = y_{i,0}^2 = y_{i,1}^2 = 0,$$

for $i = 0, 1$ and $j = 2, \dots, k$.

The relations $x_{j,i,0}^2 = x_{j,i,1}^2 = y_{i,0}^2 = y_{i,1}^2 = 0$ can be lifted to $\prod_{n=1}^{\infty} A$, thus obtaining $(x_{j,i,0}^{(n)})_{n=1}^{\infty}$, $(x_{j,i,1}^{(n)})_{n=1}^{\infty}$, and $(y_{i,j}^{(n)})_{n=1}^{\infty}$. Next, aided by Lemma 2.1, we

lift the $\phi_i(r_{j,i})$ s to selfadjoint elements $(z_{j,i}^{(n)})_n \in \prod_{n=1}^{\infty} A$, taking care that the following relations hold for all $n \in \mathbb{N}$:

$$\sum_{j=1}^k z_{j,0}^{(n)} \sim_{\text{Tr}} 0 \text{ and } z_{j,0}^{(n)} + z_{j,1}^{(n)} \sim_{\text{Tr}} h_j$$

for $j = 1, \dots, k$. (Notice that $\phi_0(r_{j,0}) + \phi_1(r_{j,1}) \sim_{\text{Tr}} h_j$ in A_{∞} .)

For each $j = 2, \dots, k$, $i = 0, 1$, and $n \in \mathbb{N}$, let us set

$$h_{j,i}^{(n)} := \sum_{k=0,1} [(x_{j,i,k}^{(n)})^*, x_{j,i,k}^{(n)}] + z_{j,i}^{(n)},$$

$$h_{1,i}^{(n)} := [(y_{i,0}^{(n)})^*, y_{i,0}^{(n)}] + [(y_{i,1}^{(n)})^*, y_{i,1}^{(n)}] - \sum_{j=2}^k z_{j,i}^{(n)}.$$

Then $(h_{j,0}^{(n)})_n$ and $(h_{j,1}^{(n)})_n$ are lifts of $\phi_0(\psi_0(h_j))$ and $\phi_1(\psi_1(h_j))$ respectively for all $j = 1, \dots, k$. In particular, they asymptotically commute. They also satisfy that $h_{j,0}^{(n)} + h_{j,1}^{(n)} \sim_{\text{Tr}} h_j$ for all n . So, applying Lemma 2.2,

$$e^{ih_j} = e^{ih_{j,0}^{(n)}} e^{ih_{j,1}^{(n)}} e^{ic_n},$$

where $c_n \sim_{\text{Tr}} 0$ for all n and $c_n \rightarrow 0$. Next, using Lemma 3.6,

$$u = \prod_{j=1}^k e^{ih_{j,0}^{(n)}} e^{ih_{j,1}^{(n)}} \cdot e^{ic'_n}$$

$$= \prod_{j=1}^k e^{ih_{j,0}^{(n)}} \prod_{j=1}^k e^{ih_{k-j,1}^{(n)}} \prod_{j=1}^{k-1} (u_{j,n}, v_{j,n}) \cdot e^{ic'_n},$$

for some $u_{j,n}, v_{j,n} \in U^0(A)$ and $c'_n \sim_{\text{Tr}} 0$ such that $c'_n \rightarrow 0$. We will show that $\prod_{j=1}^k e^{ih_{j,0}^{(n)}}$ is a product of $2k$ commutators, up to a multiplicative error of the form e^{id_n} with $d_n \sim_{\text{Tr}} 0$ and $d_n \rightarrow 0$. The same arguments will apply to $\prod_{j=1}^k e^{ih_{j,1}^{(n)}}$, thus proving the theorem.

Since the $r_{j,0}$'s are central in N_0 , $\phi_0(r_{j,0})$ commutes with $x_{j',0,0}$ and $x_{j',0,1}$ for all $j' = 2, \dots, k$. Hence, the lifts of these elements commute asymptotically. Applying Lemma 2.2 we have that

$$e^{ih_{j,0}^{(n)}} = e^{[(x_{j,0,0}^{(n)})^*, x_{j,0,0}^{(n)}]} \cdot e^{[(x_{j,0,1}^{(n)})^*, x_{j,0,1}^{(n)}]} \cdot e^{iz_{j,0}^{(n)}} \cdot e^{id_{j,n}},$$

for all $j = 2, \dots, k$, where $d_{j,n} \sim_{\text{Tr}} 0$ and $d_{j,n} \rightarrow 0$. So

$$\begin{aligned} \prod_{j=1}^k e^{ih_{j,0}^{(n)}} &= e^{ih_{1,0}^{(n)} + \sum_{j=2}^k z_{j,0}^{(n)}} \cdot \left(\prod_{j=2}^k e^{[(x_{j,0,0}^{(n)})^*, x_{j,0,0}^{(n)}]} e^{[(x_{j,0,1}^{(n)})^*, x_{j,0,1}^{(n)}]} \right) \cdot e^{id_n} \\ &= e^{i[(y_{0,0}^{(n)})^*, y_{0,0}^{(n)}]} e^{i[(y_{0,1}^{(n)})^*, y_{0,1}^{(n)}]} \left(\prod_{j=2}^k e^{[(x_{j,0,0}^{(n)})^*, x_{j,0,0}^{(n)}]} e^{[(x_{j,0,1}^{(n)})^*, x_{j,0,1}^{(n)}]} \right) \cdot e^{id'_n}. \end{aligned}$$

By Lemma 2.4, the unitaries of the form $e^{i[z^*, z]}$ in the above expression are all expressible as commutators, up to a small multiplicative error of the form e^{id} , with $d \sim_{\text{Tr}} 0$. This completes the proof. \square

Remark 3.7. When considering the possibility of extending Theorem 3.4 to C*-algebras of finite nuclear dimension $m > 1$, one stumbles with the fact that the elements of a partition of unity e_0, e_1, \dots, e_m associated to a decomposition of $\iota: A \rightarrow A_\infty$ may not pairwise commute. Thus, working in A_∞ , we cannot guarantee that for a selfadjoint $h \in A$ we have

$$(3.2) \quad e^{ih} = \prod_{j=0}^m e^{ih e_j}.$$

However, once this obstruction is lifted, it is in fact possible to make the proof of Theorem 3.4 go through. Notice that in order for (3.2) to hold, it is not necessary to ask that all the e_j s pairwise commute, but rather they the set $\{e_0, e_1, \dots, e_m\}$ can be recursively partitioned into pairs of subsets, so that the sums of the e_j s in these subsets commute. More specifically, the elements e_0, \dots, e_m are assigned in a bijective fashion to the set of leaves of some binary tree. Then, for any node of the tree λ , with children λ_0 and λ_1 , we require that the elements $\sum_{j \in \Lambda_0} e_j$ and $\sum_{j \in \Lambda_1} e_j$ commute, where Λ_0 is the set of leaves under λ_0 , and Λ_1 the leaves under λ_1 . Let us say that the elements e_0, \dots, e_m *recursively commute* if they have this property for some binary tree. Here is the statement of the resulting theorem:

Theorem 3.8. *Let A be a C*-algebra of nuclear dimension $m \in \mathbb{N}$. Suppose that A has a partition of unity $\{e_0, \dots, e_m\}$ whose elements recursively commute (in the sense described above). Let $u \in U^0(A)$ be of the form $u = e^{ih_1} e^{ih_2} \dots e^{ih_k}$, where $h_j \in A_{\text{sa}}$ for $j = 1, \dots, k$ and*

$$\sum_{j=1}^k h_j \sim_{\text{Tr}} 0.$$

Then for each $\varepsilon > 0$ there exist unitaries $v_j, w_j \in U^0(A)$, with $j = 1, \dots, M$ and $M = 2(m+1)k + \lfloor \frac{m+1}{2} \rfloor (k-1)$, such that

$$u = \prod_{j=1}^M (v_j, w_j) \cdot e^{ic},$$

for some $c \in A_{sa}$ with $c \sim_{\text{Tr}} 0$ and $\|c\| < \varepsilon$.

Proof. Let $h_{j,l} = e_l h_j$ for $j = 1, \dots, k$ and $l = 0, \dots, m$. By the recursive commutation relations of the e_l 's, we have $e^{ih_j} = \prod_{l=0}^m e^{ih_{j,l}}$. So

$$(3.3) \quad \prod_{j=1}^k e^{ih_j} = \prod_{j=1}^k \prod_{l=0}^m e^{ih_{j,l}}.$$

We use the following extension of Lemma 3.6:

$$\prod_{j=1}^k \prod_{l=0}^m a_{j,l} = \prod_{j=1}^k a_{j,0} \prod_{j=0}^{k-1} a_{k-j,1} \prod_{j=1}^k a_{j,2} \cdots \prod_{i=1}^{\lfloor \frac{m+1}{2} \rfloor (k-1)} (u_i, v_i).$$

Thus, applying this formula to (3.3),

$$\prod_{j=1}^k e^{ih_j} = \prod_{j=1}^k e^{ih_{j,0}} \prod_{j=0}^{k-1} e^{ih_{k-j,1}} \cdots \prod_{i=1}^{\lfloor \frac{m+1}{2} \rfloor (k-1)} (u_i, v_i)$$

But for each $l = 0, \dots, m$ the product of exponentials $\prod_{j=1}^k e^{ih_{j,l}}$ is expressible as a product of $2k$ commutators (as argued in the proof of Theorem 3.4). This shows that $\prod_{j=1}^k e^{ih_j}$ is a product of $M = 2(m+1)k + \lfloor \frac{m+1}{2} \rfloor (k-1)$ commutators in A_∞ . Furthermore, arguing as in the proof of Theorem 3.4, it is possible to lift these commutators so that the same expression holds in A , up to a small multiplicative error of the form e^{ic} , with $c \sim_{\text{Tr}} 0$. The details are left to the reader. \square

4. C^* -SUBALGEBRA OF NUCLEAR DIMENSION ONE IN A SIMPLE PURE C^* -ALGEBRA OF STABLE RANK ONE

In this section, we rely on classification results using the Cuntz semigroup functor to establish the following theorem:

Theorem 4.1. *Let A be a simple, σ -unital, pure, C^* -algebra of stable rank one. Then there exists a C^* -subalgebra $B \subseteq A$ with the following properties:*

- (i) $B \cong C \otimes \mathcal{R}$, where C is a simple AF C^* -algebra and \mathcal{R} is the Jacelon-Razak C^* -algebra (see [Jac13]).

- (ii) Every lower semicontinuous trace on B extends uniquely to a lower semicontinuous 2-quasitrace on A . Furthermore, for bounded traces the norm is preserved.
- (iii) Every selfadjoint $h \in A_{sa}$ with connected spectrum is approximately unitarily equivalent to some $h' \in B_{sa}$.

Before proving the theorem, we make some preliminary remarks on the Cuntz semigroup of pure C*-algebras. The reader is referred to [APT11] for the fundamental facts on the Cuntz semigroup of a C*-algebra.

Let A be a C*-algebra satisfying the hypotheses of Theorem 4.1. The Cuntz semigroup of A decomposes into two subsemigroups:

- (1) The set $\text{Cu}_c(A)$ of non-zero compact elements (i.e., $[a] \in \text{Cu}(A) \setminus \{0\}$ such that $[a] \ll [a]$) is a subsemigroup of $\text{Cu}(A)$.
- (2) The complement of $\text{Cu}_c(A)$ is also a subsemigroup which we denote by $\text{Cu}_{nc}(A)$.

Since A is stably finite, $\text{Cu}_c(A)$ is isomorphic to the semigroup $V^*(A)$ of non-zero Murray-von Neumann equivalence classes of projections (by [BC09]). On the other hand, by [ABP11, Theorem 5.6], $\text{Cu}_{nc}(A)$ is isomorphic to $\text{LAff}(\text{QT}_2(A))$, where $\text{QT}_2(A)$ is the cone of densely finite lower semicontinuous 2-quasitraces on A and $\text{LAff}(\text{QT}_2(A))$ is a suitable space of affine lower semicontinuous functions on $\text{QT}_2(A)$. We thus have

$$\begin{aligned} \text{Cu}(A) &= \text{Cu}_c(A) \sqcup \text{Cu}_{nc}(A) \\ &\cong V^*(A) \sqcup \text{LAff}(\text{QT}_2(A)), \end{aligned}$$

where $V^*(A) \sqcup \text{LAff}(\text{QT}_2(A))$ is endowed with a suitable order and addition (see [ABP11, Theorem 5.6]).

In [Rob12], the ordered semigroup $\text{Cu}^\sim(A)$ is introduced and used as a classification invariant. For A as before, it is shown in [Rob12, Section 6] that

$$\text{Cu}^\sim(A) \cong K_0(A) \sqcup \text{LAff}^\sim(\text{QT}_2(A)),$$

where again $\text{LAff}^\sim(\text{QT}_2(A))$ is a suitable ordered semigroup of lower semicontinuous functions on $\text{QT}_2(A)$. The reader is referred to [Rob12] for its definition. We point out here that $\text{LAff}(\text{QT}_2(A))$ is the positive cone of $\text{LAff}^\sim(\text{QT}_2(A))$ and that $\text{QT}_2(A)$ can be recovered from $\text{LAff}(\text{QT}_2(A))$ (and thus from $\text{LAff}^\sim(\text{QT}_2(A))$) as the cone of positive functionals.

Proof of Theorem 4.1. By [Bla80, Theorem 3.10], there exists an AF C*-algebra C such that $T(C) \cong \text{QT}_2(A)$. Tensoring with the Jacelon-Razak C*-algebra \mathcal{R} , we get a C*-algebra $B = C \otimes \mathcal{R}$ such that $K_0(B) = 0$ and

$$T(B) \cong T(C) \cong \text{QT}_2(A).$$

Thus,

$$\text{Cu}^\sim(B) \cong \text{LAff}^\sim(T(B)) \cong \text{LAff}^\sim(\text{QT}_2(A)).$$

(Up to this point, we have borrowed ideas from [Jac13, Proposition 5.3] and [Jac13, Corollary 5.4].)

Let $e_A \in A_+$ be a strictly positive element. Let $s_A \in A_+$ be such that if A is unital then $[s_A]$ is the maximal element strictly less than $[e_A]$, and if A is non-unital then $[s_A] = [e_A]$. Notice that in either case $[s_A] \in \text{Cu}_{\text{nc}}(A)$. Let us choose $[s_B] \in \text{Cu}(B)$ that is mapped to $[s_A]$ by the isomorphism $\text{Cu}(B) \cong \text{Cu}_{\text{nc}}(A)$. Passing to a stably isomorphic algebra if necessary, let us assume that $s_B \in B_+$ is a strictly positive element of B . We thus get an embedding $\text{Cu}^\sim(B) \xrightarrow{\alpha} \text{Cu}^\sim(A)$ mapping $[s_B]$ to $[s_A]$. Now observe that the C^* -algebra B is covered by the classification result of [Rob12], since it is expressible as an inductive limit of 1-dimensional NCCW-complexes with trivial K_1 -group. Thus, there exists a homomorphism $\phi: B \rightarrow A$ such that, after applying the functor Cu^\sim , gives rise to α . Since ϕ is necessarily an embedding, we will regard B as a subalgebra of A .

It is clear, by construction, that B satisfies (i). That it also satisfies (ii) follows from the fact that the inclusion $B \hookrightarrow A$ induces an isomorphism of $L(\text{QT}_2(B))$ with $L(\text{QT}_2(A))$, whence, by duality, of $\text{QT}_2(A)$ with $\text{QT}_2(B) = T(B)$.

To prove (iii) we follow essentially the argument given in [Rob13]: Say $h \in A_{\text{sa}}$ has connected spectrum. Multiplying h by a scalar if necessary, let us assume that $\|h\| \leq 1$. Let $\psi: C_0(0, 1] \oplus C_0(0, 1] \rightarrow A$ be the homomorphism such that $\psi(t \oplus 0) = h_+$ and $\psi(0 \oplus t) = h_-$. Since the spectrum of h is connected, the map $\text{Cu}(\psi)$ ranges in $\text{Cu}_{\text{nc}}(A)$. Thus, it can be factored through $\text{Cu}(B)$. By the classification theorem [CES11, Theorem 1], there exists $\psi': C_0(0, 1] \oplus C_0(0, 1] \rightarrow B$ such that $\text{Cu}(\psi') = \text{Cu}(\psi)$. Furthermore, ψ' and ψ are approximately unitarily equivalent. Thus, $h' := \psi'(t \oplus -t)$ is approximately unitarily equivalent to h . \square

5. APPROXIMATION BY COMMUTATORS

In this section we prove Theorems 5.2 and 5.3 (they are used in the proof of Theorem 1.1 from the introduction).

Lemma 5.1. *Let A be a simple non-type I C^* -algebra. Let $h \in A_{\text{sa}}$. Then for each $\varepsilon > 0$ there exist $\tilde{h} \in A_{\text{sa}}$ and $x \in A$ such that*

- (i) $h \approx_\varepsilon \tilde{h} + [x^*, x]$,
- (ii) \tilde{h} has connected spectrum,
- (iii) \tilde{h} and $[x^*, x]$ commute, $x^2 = 0$, and $\|\tilde{h}\|, \|x\| \leq \|h\|$.

Proof. Let $a = h_+$ and $b = h_-$. Let us assume that $\|a\| \geq \|b\|$ (the case $\|a\| < \|b\|$ is dealt with similarly). We first consider the case when $b \neq 0$. Using functional calculus, we can slightly perturb a (within any specified error and without changing its norm) so that $ae = \|a\|e$ for some non-zero $e \in A_+$. So

let us assume that such an e exists. Similarly, let us assume that $bf = \|b\|f$ for some non-zero $f \in A_+$. Since A is simple, we can find a non-zero $e' \in \text{her}(e)_+$ such that $e' = y^*y$ and $yy^* \in \text{her}(f)$ for some $y \in A$. Since A is also non-type I, we can find $e'' \in \text{her}(e')_+$ with spectrum $[0, \|a\|]$. Let $x \in A$ be such that $e'' = x^*x$ and $xx^* \in \text{her}(f)$. Set $\tilde{h} = h - [x^*, x]$.

Let us show that \tilde{h} and x have the desired properties. Properties (i) and (iii) are clearly true. Let us prove (ii). Since $ax^*x = \|a\| \cdot x^*x$ and the spectrum of x^*x is $[0, \|a\|]$, the spectrum of $a - x^*x$ is $[0, \|a\|]$. Similarly, since $bxx^* = \|b\| \cdot xx^*$ and the spectrum of xx^* is $[0, \|a\|]$, the spectrum of $b - xx^*$ is $[\|b\| - \|a\|, \|b\|]$. Finally, the spectrum of $\tilde{h} = (a - x^*x) - (b - xx^*)$ is $[0, \|a\|] \cup [-\|b\|, \|a\| - \|b\|] = [-\|b\|, \|a\|]$.

Let us sketch briefly how the above argument can be adapted in the case that $b = 0$: As before, we assume that $ae = \|a\|e$ for some non-zero $e \in A_+$. Next we find $x \in \text{her}(e)$ such that x^*x has spectrum $[0, \|a\|]$ and $x^2 = 0$. Finally, we set $\tilde{h} = h - [x^*, x]$. \square

Theorem 5.2. *Let A be a separable, simple, pure C*-algebra with stable rank one and such that every 2-quasitracial state on A is a trace. Let $h \in A_{\text{sa}}$, with $h \sim_{\text{Tr}} 0$ and $\|h\| < \frac{\pi}{2}$. Then for each $\varepsilon > 0$ there exist unitaries $v_j, w_j \in U^0(A)$, with $j = 1, 2, 3, 4, 5$ and $c \in A_{\text{sa}}$ such that*

$$e^{ih} = \prod_{j=1}^5 (v_j, w_j) \cdot e^{ic},$$

where $c \sim_{\text{Tr}} 0$, $\|c\| < \varepsilon$, and $\|1 - v_j\|, \|1 - w_j\| \leq \|e^{ih} - 1\|^{\frac{1}{2}}$ for all j .

Proof. Letting $\varepsilon = \frac{1}{n}$, with $n = 1, \dots$, in the previous lemma, we find $h_n \in A_{\text{sa}}$ and $x_n \in A$ such that h_n has connected spectrum, $x_n^2 = 0$, h_n and $[x_n^*, x_n]$ commute, and

$$\|h_n + [x_n^*, x_n] - h\| < \frac{1}{n}$$

for all $n \in \mathbb{N}$. Let $B \subseteq A$ be the sub-C*-algebra whose existence was established in Theorem 4.1. Then, each h_n is approximately unitarily equivalent to some $h'_n \in B$. We have $|\tau(h'_n)| = |\tau(h_n)| \leq \frac{1}{n}$ for any tracial state τ on A . Our hypotheses imply that every tracial state on B extends uniquely to a tracial state on A . Indeed, by Theorem 4.1 (ii) every tracial state on B extends to a 2-quasitracial state on A , which by assumption must be a trace. Thus, $|\tau(h'_n)| < \frac{1}{n}$ for every tracial state τ on B . We can then perturb h'_n (up to an error of $\frac{1}{n}$) inside of B to get $h''_n \in B_{\text{sa}}$ such that $h''_n \sim_{\text{Tr}} 0$ (in B). We thus have that $u_n^* h''_n u_n + [x_n^*, x_n] \rightarrow h$, where $u_n \in U(A)$ are unitaries. By Lemma 2.2 we have

$$e^{ih} = e^{i[x_n^*, x_n]} u_n^* e^{ih''_n} u_n e^{ic_n},$$

where $c_n \sim_{\text{Tr}} 0$ and $c_n \rightarrow 0$. Furthermore, by Lemma 2.4, $e^{i[x_n^*, x_n]} = (v_{n,0}, w_{n,0})$ for some $v_{n,0}, w_{n,0} \in U^0(A)$. Thus,

$$(5.1) \quad e^{ih} = (v_{n,0}, w_{n,0}) u_n^* e^{ih''} u_n e^{ic_n}.$$

Finally, applying Theorem 3.3 in the algebra B – which has nuclear dimension 1 – we get unitaries $v_{n,j}, w_{n,j} \in U^0(A)$, for $n = 1, 2, 3, 4$ such that

$$e^{ih} = (v_{n,0}, w_{n,1}) \prod_{j=1}^4 (v_{n,j}, w_{n,j}) e^{ic'_n}.$$

where $c'_n \sim_{\text{Tr}} 0$ and $c'_n \rightarrow 0$. Lemmas 5.1 and 2.4 and Theorem 3.3 also guarantee that the inequalities $\|v_{n,j} - 1\|, \|w_{n,j} - 1\| \leq \|e^{ih} - 1\|^{\frac{1}{2}}$ hold for $j = 0, 1, 2, 3, 4$ and all n . \square

Theorem 5.3. *Let A be as in Theorem 5.2. Let $u \in U^0(A)$ be of the form $u = \prod_{j=1}^k e^{ih_j}$, where $h_j \in A_{\text{sa}}$ for all j , and such that $\Delta_{\text{Tr}}(u) = 0$. Then for each $\varepsilon > 0$ there exist unitaries $v_j, w_j \in U^0(A)$, with $j = 1, \dots, 7k+6$, such that*

$$u = \prod_{j=1}^{7k+6} (v_j, w_j) \cdot e^{ic}$$

where $c \sim_{\text{Tr}} 0$ and $\|c\| < \varepsilon$.

Before proving the theorem, we need some preparatory lemmas. We let \mathcal{Z} denote the Jiang-Su C^* -algebra.

Lemma 5.4. *For each non-zero $e \in \mathcal{Z}_+$ we can find $h \in \text{her}(e)_{\text{sa}}$ such that $h \sim_{\text{Tr}} 1$ and $e^{2\pi i h} = (u, v)$ for some unitaries $u, v \in U^0(\text{her}(e))$.*

Proof. It suffices to prove the lemma for a sufficiently “Cuntz small” element e . For suppose that we are given $e \in \mathcal{Z}_+$, and that the lemma has been proven for some $e' \in \mathcal{Z}_+$ such that $e' = x^*x$ and $xx^* \in \text{her}(e)$ for some $x \in \mathcal{Z}$. Say $h' \in \text{her}(e')_{\text{sa}}$ and $u', v' \in U^0(\text{her}(e'))$ are such that $h' \sim_{\text{Tr}} 1$ and $e^{2\pi i h'} = (u', v')$. Let $w \in \mathcal{Z}^{**}$ be a partial isometry in the bidual of \mathcal{Z} such that $x = w|x|$. Then setting $h = wh'w^*$, $u = wu'w^*$ and $v = wv'w^*$, the lemma is proven for e .

Let $n \in \mathbb{N}$ and consider the dimension drop algebra $\mathcal{Z}_{n-1,n}$ as a subalgebra of \mathcal{Z} . Let $e \in \mathcal{Z}_{n-1,n}$ be a positive element such that $\text{rank}(e(t)) = n$ for $t \in (0, 1]$ and $\text{rank}(e(0)) = n-1$. Then $(n-1)[e] \leq [1] \leq n[e]$ in $\mathcal{Z}_{n-1,n}$ (whence in \mathcal{Z}). Therefore, by choosing n large enough, we can arrange for $[e]$ to be arbitrarily small in the Cuntz semigroup. By the argument in the previous paragraph, it suffices to prove the lemma for such an e . The hereditary subalgebra in $\mathcal{Z}_{n-1,n}$

generated by e is isomorphic to

$$A_{n-1,n} = \left\{ f \in M_n(C[0,1]) \mid \begin{array}{l} f(0) = \mu 1_{n-1} \\ f(1) = \lambda 1_n \end{array}, \text{ for some } \lambda, \mu \in \mathbb{C} \right\}.$$

Let $\omega: [0,1] \rightarrow [0,1]$ be a continuous function such that $\omega(t) = 0$ for $t \in [0, \frac{1}{3}]$, ω is linear in $[\frac{1}{3}, \frac{2}{3}]$, and $\omega(t) = 1$ for $t \in [\frac{2}{3}, 1]$. Let

$$h(t) = \begin{pmatrix} n - \omega(t) & & & \\ & n - \omega(t) & & \\ & & \ddots & \\ & & & (n-1)\omega(t) \end{pmatrix}.$$

Notice that $h \in A_{n-1,n}$ and that identifying $A_{n-1,n}$ with $\text{her}(e) \subseteq \mathcal{Z}_{n-1,n}$ we have $h \sim_{\text{Tr}} 1 \in \mathcal{Z}_{n-1,1}$. Hence, $h \sim_{\text{Tr}} 1$ in \mathcal{Z} after identifying $\mathcal{Z}_{n-1,n}$ with a unital subalgebra of \mathcal{Z} . We will be done once we have shown that $e^{2\pi i h} = (u, v)$, for some $u, v \in U_0(A_{n-1,n})$.

We have

$$e^{2\pi i h(t)} = \begin{pmatrix} e^{-2\pi i \omega(t)} & & & \\ & e^{-2\pi i \omega(t)} & & \\ & & \ddots & \\ & & & e^{2\pi i (n-1)\omega(t)} \end{pmatrix}.$$

Notice that $e^{ih(t)}$ is the identity matrix for $t \in [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$ and that the elements on the diagonal of $e^{ih(t)}$ multiply to 1 for all $t \in [0, 1]$. So $e^{ih(t)} = (P, v(t))$ for all $t \in [\frac{1}{3}, \frac{2}{3}]$, where P is a suitable permutation unitary and $v(t) \in M_n$ is a diagonal unitary for all $t \in [\frac{1}{3}, \frac{2}{3}]$ (see by [Tho93, Lemma 2.1]). Furthermore, $v(\frac{1}{3}) = v(\frac{2}{3}) = 1_n$. Thus, after extending v to be constant equal to 1_n on $[0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$ and extending P using a path that connects it to the identity matrix 1_n we get $u, v \in U^0(A_{n-1,n})$ such that $e^{ih} = (u, v)$. \square

Lemma 5.5. *Let A be a pure, simple, unital C*-algebra with stable rank one. Let $p \in M_n(A)$ be a projection in some matrix algebra over A . For each $e \in A_+$ we can find a selfadjoint $h \in \text{her}(e)_+$ such that $h \sim_{\text{Tr}} p$ and $e^{ih} = (v, w)$ for some $v, w \in U_0(A)$.*

Proof. By the classification theorem from [Rob12], we can embed \mathcal{Z} unitaly in $pM_n(A)p$. Let us choose $e \in \mathcal{Z}_+$ such that $(n+1)[e] \leq [p]$ in \mathcal{Z} . By the previous lemma, there exists $h \in \text{her}(e)_{\text{sa}} \cap \mathcal{Z}$ such that $h \sim_{\text{Tr}} p$ (inside \mathcal{Z}) and $e^{ih} = (u, v)$ for some $u, v \in U^0(\text{her}(e) \cap \mathcal{Z})$. Notice that $h \sim_{\text{Tr}} p$ also in $M_n(A)$. Since $(n+1)[e] \leq [p] \leq n[1]$ and $\text{Cu}(A)$ is almost unperforated, we have $[e] \leq [1]$ in $\text{Cu}(A)$. Furthermore, A has stable rank one, so there exists $x \in M_n(A)$ such that $e = x^*x$ and $xx^* \in A$. Let $w \in (M_n(A))^{**}$ be a partial

isometry such that $x = w|x|$. Setting $h = wh'w^*$, $u = wu'w^*$ and $v = wvw^*$, we get the desired elements in A . \square

Remark 5.6. The previous proof can be adapted to also apply to any \mathcal{Z} -stable C^* -algebra.

Lemma 5.7. *Let A be a separable simple C^* -algebra of stable rank one. Let $u \in U(A)$ and $h \in A_{sa}$. For any $\varepsilon > 0$ there exist $v, w \in U^0(A)$ such that*

$$(u, e^{ih}) = (v, w)e^{ic},$$

for some $c \in A_{sa}$ such that $c \sim_{\text{Tr}} 0$ and $\|c\| < \varepsilon$.

Proof. By functional calculus on h , we can find $e \in A_+$ such that $\|[h, x]\| \leq \delta\|x\|$ for any $x \in \text{her}(e)$, where $\delta > 0$ is small enough (how much to be specified soon). Since A and $\text{her}(e)$ are stably isomorphic and have stable rank one, $U(A)/U^0(A) \cong K_1(A) \cong U(\text{her}(e))/U^0(\text{her}(e))$. Thus, there exists $v \in U(\text{her}(e)) \subseteq U(A)$ connected to u . Choosing δ small enough, we can arrange that $\|\text{Log}(ve^{ih}v^*e^{-ih})\| < \varepsilon$, so that $ve^{ih}v^* = e^{ih}e^{ic}$, for some $c \sim A_{sa}$ with $c \sim_{\text{Tr}} 0$ and $\|c\| < \varepsilon$. Then,

$$(u, e^{ih}) = uv^*(ve^{ih}v^*)vu^*e^{-ih} = uv^*e^{ih}e^{ic}vu^*e^{-ih} = (uv^*, e^{ih})e^{ic'},$$

where $c' \sim_{\text{Tr}} 0$ and $\|c'\| < \varepsilon$. Since $uv^* \in U^0(A)$, the lemma is proven. \square

Proof of Theorem 5.3. Since $\Delta_{\text{Tr}}(u) = 0$, there exist $n \geq \mathbb{N}$ and projections $p, q \in M_n(A)$ such that

$$\sum_{j=1}^k h_j + p - q \sim_{\text{Tr}} 0.$$

Hence, by Lemma 5.5, there exist $h_{k+1} \in A_{sa}$ and $u_0, v_0 \in U^0(A)$ such that

$$u = \prod_{j=1}^{k+1} e^{ih_j}(u_0, v_0)$$

and

$$\sum_{j=1}^{k+1} h_j \sim_{\text{Tr}} 0.$$

As in the proof of Theorem 5.2 (see (5.1)), we find that for each $j = 1, \dots, k+1$ there exist $u_j \in U(A)$, $h'_j \in B_{sa}$, and $v_j, w_j \in U^0(A)$ such that

$$e^{ih_j} = (v_j, w_j)u_j^*e^{ih'_j}u_je^{ic_j},$$

for some $c_j \in A_{sa}$ such that $c_j \sim_{\text{Tr}} 0$ and $\|c_j\|$ is small. We thus get that

$$\begin{aligned} u &= \prod_{j=1}^{k+1} u_j^* e^{ih'_j} u_j \prod_{j=1}^{k+2} (v'_j, w'_j) e^{ic} \\ &= u_1^* \left(\prod_{j=1}^{k+1} e^{ih'_j} \right) u_1 \prod_{j=1}^k (v''_j, w''_j) \prod_{j=1}^{k+2} (v'_j, w'_j) e^{ic'}. \end{aligned}$$

Here we have used that $(e^{ih'_j}, u_j) = (v''_j, w''_j) e^{ic'_j}$, with $v''_j, w''_j \in U^0(A)$, $c'_j \sim_{\text{Tr}} 0$ and $\|c'_j\|$ small, by Lemma 5.7. Since B has nuclear dimension one, we can apply Theorem 3.4 to $\prod_{j=1}^{k+1} e^{ih'_j} \in U^0(B)$:

$$\prod_{j=1}^{k+1} e^{ih'_j} = \prod_{j=1}^{5k+4} (v'''_j, w'''_j) e^{ic''},$$

with $c'' \sim_{\text{Tr}} 0$ and $\|c''\|$ small. Hence

$$\prod_{j=1}^k e^{ih_j} = \prod_{j=1}^{5k+4} (v'''_j, w'''_j) \prod_{j=1}^k (v''_j, w''_j) \prod_{j=1}^{k+2} (v'_j, w'_j) e^{ic'''},$$

where $c''' \sim_{\text{Tr}} 0$. Choosing the elements c_j , c'_j and c'' along the way small enough we can arrange for $\|c'''\| < \varepsilon$. This proves the theorem. \square

6. EXACT PRODUCT OF COMMUTATORS

Here we prove Theorem 6.1 below and then use it in combination with the results from the previous section to prove Theorem 1.1 from the introduction.

Theorem 6.1. *There exist $\delta > 0$ and $C > 0$ such that if A is a separable, simple, pure, C*-algebra with stable rank 1 and such that every 2-quasitracial state on A is a trace, and $u \in U^0(A)$ is such that $\|u-1\| < \delta$, and $\text{Log}(u) \sim_{\text{Tr}} 0$, then*

$$(6.1) \quad u = \prod_{i=1}^{23} (v_i, w_i),$$

where $v_i, w_i \in U^0(A)$ are such that

$$(6.2) \quad \|v_i - 1\|, \|w_i - 1\| \leq C \|e^{ih} - 1\|^{\frac{1}{2}}$$

for all $i = 1, \dots, 23$.

Before proving Theorem 6.1 we need a number of lemmas. Throughout this section A denotes a C*-algebra that is separable, simple, pure, of stable rank 1, and whose bounded 2-quasitraces are traces.

Lemma 6.2. *Let $h \in A_{sa}$ be such that $h \sim_{\text{Tr}} 0$ and $\|h\| < \frac{\pi}{2}$. Then there exist $u, v \in U^0(A)$ and $h_1, h_2 \in A_{sa}$ such that $h_1, h_2 \sim_{\text{Tr}} 0$,*

$$e^{ih} = (u, v)e^{ih_1}e^{ih_2},$$

h_1 and h_2 are non-invertible, $\|u - 1\|, \|v - 1\| \leq \|e^{ih} - 1\|^{\frac{1}{2}}$, and $\|h_j\| \leq 2\|h\|$ for $j = 1, 2$.

Proof. We will assume that A is unital and h is invertible, since otherwise there is nothing to prove. Let $a = h_+$ and $b = h_-$. Since A is unital, stably finite, and its bounded 2-quasitraces are traces, there is at least one non-trivial bounded trace on A . This implies that $a \neq 0$ and $b \neq 0$. We claim that there exists $x \in \text{her}(a)$ such that $x^2 = 0$, $\|x\| \leq \|a\|$, a and $[x^*, x]$ almost commute, and $a - x^*x + xx^* - b$ is not invertible. Before proving the existence of such an x , let us show how it suffices to prove the lemma: By Lemma 2.4 (ii), there exist $u, v \in U^0(A)$ such that $e^{i[x^*, x]} = (u, v)$. Also, if a and $[x, x^*]$ commute sufficiently, then by Lemma 2.2 there exists $c \in \text{her}(a)_{sa}$ such that $c \sim_{\text{Tr}} 0$ and $e^{i[x, x^*]}e^{ia} = e^{ic}e^{i[x, x^*] + ia}$. Thus,

$$\begin{aligned} e^{ih} &= (u, v)e^{i[x, x^*]}e^{ia}e^{-ib} \\ &= (u, v)e^{ic}e^{i[x, x^*] + ia}e^{-ib} \\ &= (u, v)e^{ic}e^{i([x, x^*] + a - b)}. \end{aligned}$$

Since $bc = 0$ and $b \neq 0$, c is non-invertible. By assumption $[x, x^*] + a - b$ is non-invertible, and clearly $[x, x^*] + a - b \sim_{\text{Tr}} a - b \sim_{\text{Tr}} 0$. Let us set $h_1 = c$ and $h_2 = [x, x^*] + a - b$, so that $e^{ih} = (u, v)e^{ih_1}e^{ih_2}$, with h_1 and h_2 non-invertible. The norm estimates in the statement of the lemma are easily checked. The lemma is thus proved.

Let us prove the existence of x as above. Let $\varepsilon > 0$ (which will measure how much a and $[x^*, x]$ commute). Let us first assume that the spectrum of a has at least two distinct non-zero points $0 < t_1 < t_2 \leq \|a\|$. Let us choose functions $g_1, g_2 \in C_0(0, 1]_+$ supported in small intervals around t_1 and t_2 respectively, so that the nonzero elements $e_1 = g_1(a)$ and $e_2 = g_2(a)$ satisfy that $e_1 \perp e_2$ and $\|[a, y]\| \leq \varepsilon\|y\|$ for any $y \in \text{her}(e_i)$ and $i = 1, 2$. Now, since A is simple, there exists $y \in A$ such that e_2ye_1 is nonzero. Applying some functional calculus to $|e_2ye_1| \in \text{her}(e_1)_+$, we can find non-zero contractions $f, f' \in \text{her}(e_1)_+$ and $x \in A$ such that $f'f = f$, $a^{\frac{1}{2}}fa^{\frac{1}{2}} = x^*x$, and $xx^* \in \text{her}(e_2)$. Notice that $x^2 = 0$, $\|x\| \leq \|a\|$, and $[x^*, x]$ and a commute within an error of $2\varepsilon\|a\|$. Let us show that $a - x^*x + xx^* - b$ is not invertible. Recall that we have assumed that $h = a - b$ is invertible. This implies that a is invertible in $C^*(a)$. Let

$d \in C^*(a)$ denote the inverse of $a^{1/2}$ inside $C^*(a)$. Then

$$\begin{aligned} f'd(a - x^*x + xx^* - b) &= f'a^{1/2} - f'fa^{\frac{1}{2}} + f'dxx^* \\ &= f'dxx^* \\ &= 0. \end{aligned}$$

Here we have used that $f'd \in \text{her}(e_1)$, $xx^* \in \text{her}(e_2)$, and $e_1 \perp e_2$. Thus, $f'd \perp a - x^*x + xx^* - b$, while $f'd \neq 0$. Thus, $a - x^*x + xx^* - b$ is not invertible in A .

If a has only a non-zero point in its spectrum then it is a scalar multiple of a projection. In this case, it suffices to choose $x \in \text{her}(a)$ such that $x^2 = 0$ and $\|x\| = \|a\|$. The details are left to the reader. \square

Lemma 6.3. *Let $h \in A_{sa}$ be non-invertible. Then there exists $e \in A_+$ such that $h \in \text{her}(e)$ and 0 is not an isolated point of the spectrum of e .*

Proof. If 0 is not an isolated point of the spectrum of $|h|$, we are done. Otherwise, let p be the support projection of $|h|$. Since h is not invertible we have $p < 1$. So $(1-p)A(1-p)$ is a non-type I simple C*-algebra, which implies that there exists a positive element $e' \in (1-p)A(1-p)$ such that 0 is not an isolated point of its spectrum. Then $e = p + e'$ has the desired property. \square

The following lemma is a consequence of [dlHS84b, Lemma 5.17] and [dlHS84b, Proposition 5.18].

Lemma 6.4. *Let $e, f \in A_+$ be such that $e \perp f$ and $\overline{eA} \cong \overline{fA}$ as right Hilbert A -modules.*

- (i) *Let $g \in \text{her}(e)_{sa}$. Then there exist $u, v \in \text{her}(e + f)$ and $g' \in \text{her}(f)_{sa}$ such that $e^{ig} = (u, v)e^{ig'}$, with $\|u - 1\|, \|v - 1\| \leq \|e^{ig} - 1\|$, $g' \sim_{\text{Tr}} g$ and $\|g'\| = \|g\|$.*
- (ii) *Let $h \in \text{her}(e + f)_{sa}$ be such that $h \sim_{\text{Tr}} 0$ and $\|e^{ih} - 1\| < \frac{1}{4}$. Then there exist $u_i, v_i \in U^0(\text{her}(e + f))$, with $i = 1, 2$, and $h' \in \text{her}(e)_{sa}$, such that*

$$e^{ih} = (u_1, v_1)(u_2, v_2)e^{ih'}$$

and

$$\begin{aligned} \|e^{ih'} - 1\| &\leq 8\|e^{ih} - 1\|, \\ \|u_1 - 1\|, \|v_1 - 1\| &\leq 2\|e^{ih} - 1\|^{\frac{1}{2}}, \\ \|u_2 - 1\|, \|v_2 - 1\| &\leq 2\|e^{ih} - 1\|^{\frac{1}{2}}. \end{aligned}$$

Furthermore, if $\|h\|$ is small enough then $h' \sim_{\text{Tr}} 0$.

Proof. Let $p, q \in M(\text{her}(e + f))$ be the multiplier support projections of e and f respectively. The hypothesis that $\overline{eA} \cong \overline{fA}$ implies that they are Murray-von

Neumann equivalent in $M(\text{her}(e+f))$. Now (i) follows by a direct application of [dlHS84b, Lemma 5.17] while (ii) follows from [dlHS84b, Proposition 5.18]. \square

Before proving the next lemma, we introduce some notation. Let $e \in A_+$ be such that 0 is not an isolated point of the spectrum of e . Then $[e]$ is a non-compact element of $\text{Cu}(A)$. By the computation of $\text{Cu}(A)$ (see Section 4), there exist Cuntz classes $[e_1], [e_2], \dots$ such that $[e] = 2[e_1]$ and $[e_n] = 2[e_{n+1}]$ for all $n \in \mathbb{N}$. Since A has stable rank one, Cuntz equivalence is the same as Hilbert module isomorphism (by [CEI08, Theorem 3]). Using this, we can choose the representatives $e_n \in A_+$ of these Cuntz classes such that

- (1) $\overline{eA} = \overline{e_1A + e_2A + \dots}$,
- (2) $e_n \perp e_m$ for all $n, m \geq 1$ with $n \neq m$.

By construction, we also have the (right) Hilbert A -module isomorphisms

- (1) $\overline{eA} \cong \overline{e_1A} \oplus \overline{e_1A}$,
- (2) $\overline{e_nA} \cong \overline{e_{n+1}A} \oplus \overline{e_{n+1}A}$ for all $n \in \mathbb{N}$.

In proving the following lemma we use a technique that goes back to [Fac82]. We let $e, e_1, e_2, \dots \in A_+$ be as in the previous paragraph.

Lemma 6.5. *There exist $\delta > 0$ and $C > 0$ (not depending on the C^* -algebra A) such that if $h \in \text{her}(e)_{\text{sa}}$ is such that $h \sim_{\text{Tr}} 0$ and $\|h\| < \delta$, then*

$$e^{ih} = \prod_{j=1}^{11} (v_j, w_j),$$

for some $v_j, w_j \in U^0(A)$, $j = 1, \dots, 11$, such that

$$(6.3) \quad \|v_j - 1\|, \|w_j - 1\| \leq C \|e^{ih} - 1\|^{\frac{1}{2}}$$

for all $j = 1, \dots, 11$.

Proof. By Lemma 6.4 (ii), there exist unitaries $u_1, v_1, u_2, v_2 \in U^0(A)$ such that

$$(6.4) \quad e^{ih} = (u_1, v_1)(u_2, v_2)e^{ih_1},$$

where $h_1 \in \text{her}(e_1)_{\text{sa}}$ and $h_1 \sim_{\text{Tr}} 0$. We will show that e^{ih_1} is a product of 9 commutators.

For $n = 1, 2, \dots$ let $e'_{n+1} \in \text{her}(e_n)_+$ be a positive element such that $\overline{e'_{n+1}A} \cong \overline{e_{n+1}A}$.

By Theorem 5.2 applied in $\text{her}(e_1)$, there exist unitaries $u_i^{(1)}, v_i^{(1)}$, with $i = 1, \dots, 5$, in $U^0(\text{her}(e_1))$ such that

$$e^{ih_1} = \prod_{i=1}^5 (u_i^{(1)}, v_i^{(1)})e^{ih'_1},$$

with $h'_1 \in \text{her}(e_1)_{sa}$, $h'_1 \sim_{\text{Tr}} 0$, and $\|h'_1\|' < \frac{1}{2}$. Next, by Lemma 6.4 (ii), there exist unitaries $u_6^{(1)}, v_6^{(1)}, u_7^{(1)}, v_7^{(1)}$ in $U^0(\text{her}(e_1))$ such that

$$e^{ih'_1} = (u_6^{(1)}, v_6^{(1)})(u_7^{(1)}, v_7^{(1)})e^{ih''_1},$$

and $h''_1 \in \text{her}(e'_2)_{sa}$ and $h''_1 \sim_{\text{Tr}} 0$. Finally, by Lemma 6.4 (i), we have

$$e^{ih'_1} = (y^{(1)}, z^{(1)})e^{ih_2},$$

with $y^{(1)}, z^{(1)} \in U^0(\text{her}(e_1 + e_2))$ and $h_2 \in \text{her}(e_2)_{sa}$. Next, we apply again Theorem 5.2 in $\text{her}(e_2)$:

$$e^{ih_2} = \prod_{i=1}^5 (u_i^{(2)}, v_i^{(2)})e^{ih'_2},$$

with $h'_2 \sim_{\text{Tr}} 0$ and $\|h'_2\| < \frac{1}{2^2}$, followed by Lemma 6.4 (ii),

$$e^{ih'_2} = (u_6^{(2)}, v_6^{(2)})(u_7^{(2)}, v_7^{(2)})e^{ih''_2},$$

with $h''_2 \in \text{her}(e'_3)_{sa}$ and $h''_2 \sim_{\text{Tr}} 0$, and Lemma 6.4 (ii):

$$e^{ih''_2} = (y^{(2)}, z^{(2)})e^{ih_3},$$

with $h_3 \in \text{her}(e_3)_{sa}$ and $h_3 \sim_{\text{Tr}} 0$. Continuing this strategy, we construct for each $n \in \mathbb{N}$ unitaries $u_1^{(n)}, v_1^{(n)}, \dots, u_7^{(n)}, v_7^{(n)} \in U^0(\text{her}(e_n))$, $y^{(n)}, z^{(n)} \in U^0(\text{her}(e_n + e_{n+1}))$, and a selfadjoint $h_n \in \text{her}(e_n)_{sa}$, such that $h_n \sim_{\text{Tr}} 0$, $h_n \rightarrow 0$, and

$$e^{ih_1} = \prod_{k=1}^{n-1} \left(\prod_{j=7}^j (u_j^{(k)}, v_j^{(k)}) \right) (y^{(k)}, z^{(k)}) \cdot e^{ih_n}.$$

Since $h_n \rightarrow 0$, the above formula yields e^{ih_1} expressed as an infinite product of commutators. By the pairwise orthogonality of the e_n s, we can gather the terms of this infinite product into subsequences, each of them equal to a finite product of commutators, as follows:

- (1) $\prod_{k=1}^{\infty} \prod_{j=7}^j (u_j^{(k)}, v_j^{(k)})$ is a product of 7 commutators,
- (2) $\prod_{k=1}^{\infty} (y^{(2k-1)}, z^{(2k-1)})$ is a single commutator,
- (3) $\prod_{k=1}^{\infty} (y^{(2k)}, z^{(2k)})$ is a single commutator.

We thus arrive at an expression of e^{ih_1} as a product of 9 commutators. This, together with (6.4), yields an expression of e^{ih} as a product of 11 commutators. The estimates (6.3) are straightforwardly derived from the construction of the unitaries involved in the commutators. \square

Proof of Theorem 6.1. Let $h \in A_{sa}$ be such that $\|h\| < \frac{\pi}{2}$. Then by Lemma 6.2, we have $e^{ih} = (u, v)e^{ih_1}e^{ih_2}$, where $h_1, h_2 \in A_{sa}$ are non-invertible and $h_1, h_2 \sim_{\text{Tr}} 0$. (If A is non-unital this step is unnecessary). But e^{ih_1} and e^{ih_2}

are both products of 11 commutators, by Lemma 6.5. This yields (6.1). The verification of the norm estimates (6.2) is left to the reader. \square

Proof of Theorem 1.1. By Theorem 5.3, we have that

$$u = \prod_{j=1}^{7k+6} (v_j, w_j) \cdot e^{ic},$$

where $v_j, w_j \in U^0(A)$ and $c \in A_{sa}$, with $c \sim_{\text{Tr}} 0$ and $\|c\|$ as small as desired. Thus, we can arrange for Theorem 6.1 to be applicable, so that e^{ic} is expressed as a product of 23 commutators. In this way, u is expressed as a product of $7k + 6 + 23 = 7k + 29$ commutators. \square

Corollary 6.6. *Let A be as before. Then*

$$\text{DU}^0(A) = \text{DU}(A) = \ker(\Delta_{\text{Tr}}).$$

Proof. The equality $\text{DU}^0(A) = \ker(\Delta_{\text{Tr}})$ is a corollary of the previous theorem together with the fact that every element of $U^0(A)$ is a finite product of exponentials.

To prove $\text{DU}(A) = \ker(\Delta_{\text{Tr}})$, it suffices to show that $\text{DU}(A) \subseteq \ker(\Delta_{\text{Tr}})$ (since we already know that $\text{DU}^0(A) = \ker(\Delta_{\text{Tr}})$). In general,

$$\text{DU}(A) \cap U^0(A) \subseteq \ker(\Delta_{\text{Tr}}).$$

To see this, notice that elements of $\text{DU}(A)$ are in $\text{DU}^0(M_3(A))$, since

$$\text{diag}((u, v), 1, 1) = (\text{diag}(u, u^*, 1), (\text{diag}(v, 1, v^*))).$$

Hence, if an element of $\text{DU}(A)$ is in $U^0(A)$, then its de la Harpe-Skandalis determinant evaluates to 0. Since in our case A is assumed to have stable rank one, $\text{DU}(A) \subseteq U^0(A)$ by K_1 -injectivity. Thus, $\text{DU}(A) \subseteq \ker(\Delta_{\text{Tr}})$. \square

Theorem 6.7. *Let A be a separable, simple, pure C^* -algebra with stable rank 1 and such that every 2-quasitracial state on A is a trace. Let $x \in \text{GL}^0(A)$.*

If $\Delta_{\text{Tr}}(x) = 0$ then x is a finite product of multiplicative commutators in $\text{GL}^0(A)$.

Sketch of proof. The proof is very similar to the proof in the unitary case. We briefly indicate the modifications which need to be made.

Step 1: We establish the following: There exists a number $k \geq 1$ such that if $a \in A_{sa}$ and $\text{Tr}(a) = 0$ then e^a is a product of k multiplicative commutators.

The proof of Step 1 is very similar to the proof of Lemma 6.5. Firstly, we prove a version of Theorem 5.2 for exponentials of the form e^c with $c \in A_{sa}$ (as opposed to unitary exponentials). The proof is exactly the same as that for Theorem 5.2, except that in the proof of Lemma 2.4, [dlHS84b] Lemma 5.13 needs to be replaced with [Tho93] Lemma 2.6. Then we follow exactly

the same line of argument as that of Lemma 6.5, except that Lemma 6.4 needs to be replaced with [dlHS84b] Lemma 5.11 and Proposition 5.12. (In other words, we are actually following the line of argument of [dlHS84b] Proposition 6.1.)

Step 2: Now suppose that $x \in GL^0(A)$ with $\Delta_{\text{Tr}}(x) = 0$. Let $x = u|x|$ be the polar decomposition of x . Then $\Delta_{\text{Tr}}(u) = \Delta_{\text{Tr}}(|x|) = 0$.

Since $\Delta_{\text{Tr}}(u) = 0$, by Theorem 1.1, u is a finite product of commutators. Since $|x|$ is a positive invertible, $\Delta_{\text{Tr}}(|x|) = 0$ implies that $\text{Tr}(\text{Log}(|x|)) = 0$. Hence, by Step 1, $|x|$ is a finite product of commutators. Hence, x is a finite product of commutators, as required. □

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